Entropy Production in a Persistent Random Walk

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We consider a one-dimensional persisent random walk viewed as a deterministic process with a form of time reversal symmetry. Particle reservoirs placed at both ends of the system induce a density current which drives the system out of equilibrium. The phase space distribution is singular in the stationary state and has a cumulative form expressed in terms of generalized Takagi functions. The entropy production rate is computed using the coarse-graining formalism of Gaspard, Gilbert and Dorfman. In the continuum limit, we show that the value of the entropy production rate is independent of the coarse-graining and agrees with the phenomenological entropy production rate of irreversible thermodynamics.
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I. INTRODUCTION

One of the most interesting and, at the moment, controversial problems in non-equilibrium statistical mechanics is to understand the mircoscopic origins of the entropy production in non-equilibrium processes, especially the entropy production envisaged by irreversible thermodynamics for the usual hydrodynamic processes. By this one means that one should start from some totally microscopic description of a system in some intitial state, show that in the course of time the system approaches a stationary state, in some sense, and then calculate the entropy production associated with this process. There is no clear definition of the microscopic entropy production, however, and this in itself presents a problem. One choice of an entropy was provided by Gibbs with a definition based upon the full phase space distribution function of a classical system. However, the Gibbs entropy defined with respect to the phase space distribution function remains constant in time, if the time dependence of the distribution function is determined by the Liouville equation, as it is for conservative, Hamiltonian systems. One solution to this particular problem has long been discussed, the use of the Gibbs entropy as a measure of a non-equilibrium entropy requires that every possible trajectory in phase space (except for a set of measure zero) be followed in time with infinite precision, but if one relaxes this requirement and only follows trajectories to within some specified precision, i.e. "coarse grains" the description, then one obtains an entropy function that increases with time. Of course, an increase with time does not automatically imply an agreement of the entropy production with the laws of irreversible thermodynamics.

As a consequence of these considerations, a number of issues remain to be resolved:

1. Is a coarse grained Gibbs entropy the best candidate for a definition of a non-equilibrium entropy ?
2. If so, how does one correctly define the coarse graining process ? To what extent are results so obtained independent of the coarse graining procedure ?
3. Do any definitions of entropy production lead to the laws of irreversible thermodynamics, and if so, when and why ?

There is, of course an enormous literature on all of these questions. Here we wish only to discuss some recent progress in answering them based upon the approach to a theory of irreversible processes through dynamical systems theory. Gaspard [6] has considered a microscopic model of diffusion of particles in one dimension, called a multi-baker model, where the dynamics is modeled by a baker's transformation taking place on a one-dimensional lattice, where a unit square is associated to every site. Here the one-dimensional lattice is identified as the configuration space of a random walker where diffusion takes place. The baker's transformation exchanges points of the unit squares between neighboring cells and allows for a deterministic description of the random walk, i. e. keeping the dynamics on the configuration space unchanged. The variables on the unit square are irrelevant to the diffusion process and,

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in themselves, have no physical meaning, other than insuring the measure-preserving nature and reversibility of the dynamics. Gaspard considered a finite length chain, $1 \leq n \leq L$, and defined the dynamics on the unit squares so that particles would be sent either to the adjacent right or left intervals depending upon their location along the expanding direction in the unit square. Tasaki and Gaspard [14] considered the case where a steady gradient in particle density was maintained along the chain, and were able to show that fractal-like structures formed by regions of differing microscopic densities, appear in the two dimensional phase space. The fractal like structures become real fractals in the infinite volume limit, as $L \to \infty$.

These fractal-like structures are strict consequences of the dynamics given the presence of a density gradient produced by particle reservoirs at the boundaries. Their importance for the theory of entropy production, as pointed out by Gaspard [7,8], lies in the fact that they provide a fundamental reason for coarse graining the distribution function. That is, for large systems, the microscopic variations in density take place on such fine scales that no reasonable measurement process would or should be able to detect these variations. Gaspard [7,8] showed that the steady state production of entropy in this model is in agreement with the predictions of irreversible thermodynamics and further, that the entropy production is independent of the coarse graining size over a wide range of possible coarse graining sizes. While it is not entirely clear why this procedure leads to results in agreement with irreversible thermodynamics, the model and procedure are sufficiently interesting and stimulating that one would hope to find further examples so as to gain some deeper insights into the nature of entropy production, hopefully in general, and certainly in this group of baker transformation like models.

A closely related, independent approach to the problem of entropy production in non-equilibrium steady states is provided by Tél, Vollmer, and Breymann (TVB) in a series of papers [1,19,2,20], also devoted to diffusion in multi-baker models. These authors considered the entropy production in measure preserving maps as well as in dissipative maps that do not preserve the Lebesgue measure and model systems with Gaussian thermostats. The TVB systems also show that a coarse grained distribution function leads to a positive entropy production in agreement with non-equilibrium thermodynamics. In addition to considering a wider class of models than Gaspard, they used a different coarse graining scheme that was not devised to expose the underlying fractal structures of the SRB measures of the two dimensional phase space associated with their models. They also argued that their results for the entropy production should be largely independent of the type of coarse graining scheme used.

A generalization of the methods of Gaspard [7,8] and TVB [1,19,2,20] was proposed by the present authors in a recent paper [9], where it was shown that a coarse-grained form of the Gibbs entropy, which can be expressed in terms of the measures and volumes of the coarse graining sets partitioning the phase, leads to an entropy production formula similar to Gaspard's and applicable to more general volume-preserving as well as dissipative models, such as those considered by TVB, as well as multi-baker maps with energy flow considered by Tasaki and Gaspard [15].

We should also mention that multi-baker models and the use of coarse gaining methods for calculating their entropy production have been criticized by Rondoni and Cohen [12]. While some of their points are indisputably correct, their approach does not suggest a better way to proceed. So we continue this line of research in the expectation that it will lead to some further insights into entropy production in more realistic models, despite the shortcomings of the simplified models we treat here.

In this paper, we extend the ideas mentioned above, particularly those of Gaspard [7,8] and Gilbert and Dorfman [9], to a somewhat more complex model than those considered above. We consider a deterministic version of a persistent random walk in one dimension. The persistent random walk (PRW) is similar to the usual random walk , but in this case the moving particle has both a position and velocity as it moves along a one dimensional lattice. At each lattice site the particle encounters a scatterer which, with probability p allows the particle to continue in the direction of its velocity and with probability $q = 1 - p$ reverses the direction of the velocity. This model is a limiting version of a Lorentz lattice gas described in detail by van Velzen and Ernst [18], where scatterers are distributed at random along the lattice sites with some overall density per site. In the persistent random walk, the site density is unity. Although this is clearly a random process, it can be turned into a deterministic one by a method described by Dorfman, Ernst, and Jacobs [4], based upon the baker's transformation, whereby new variables are added to the system such that the dynamics in terms of these new variables allows us to replace the stochastic scattering mechanism by a deterministic one. A similar mechanism has also been considered by Goldstein, Lanford and Lebowitz in a different context [10]. Further, we can place many particles on the lattice as long as they do not interact with each other and if we take each individual scattering event to be independent of any others taking place at the same instant of time.

Here we will consider the PRW in one dimension as a model for diffusion and entropy production, and we will describe the entropy production in terms of the additional phase space variables needed to make the system deterministic. That an irreversible entropy production is to be associated with this process follows from the fact that for systems with periodic boundary condiditons, any intitial variations in the probability of finding a particle at given positions will eventually vanish and the probability will become uniform with time. Thus information is lost about the position of the particle and the entropy of the system is thereby increased. It is of some interest to see how this loss of information is reflected in the phase space distribution function, and how the entropy so produced is related to the

entropy which is the subject of irreversible thermodynamics. We will see that the analysis of the PRW model has some features that one hopes would be more general, such as the clear independence of the entropy production on the sizes of the coarse graining regions, provided these regions are not too small, and the coincidence of the production of the coarse grained Gibbs entropy with the entropy production of irreversible thermodynamics, previously noted by Gaspard, TVB and the present authors in multi-baker maps.

The organization of this paper is as follows. Section II gives a simple description of the phenomenological approach to entropy production in a simple one-dimensional diffusive system. In Sec. III, we describe in more detail the PRW process. In Sec. IV we describe a two-dimensional, measure preserving map that reproduces the PRW on a macroscopic scale and discuss its time-reversal properties. We consider a system of non-interacting particles with a steady density gradient, produced by particle reservoirs at each end of the system. We then obtain the steady state invariant measure using techniques due to Gaspard and Tasaki [14]. In Sec. V we develop a symbolic dynamics for the diffusion process, as it is refected in phase space, and, in Sec. VI, we apply this symbolic dynamics to compute the rate of entropy production. We conclude with remarks and a discussion of open questions in Sec. VII.

II. PHENOMENOLOGICAL APPROACH

Before turning to the description of the persistent random walk in Sec. III, we briefly discuss the phenomenological approach to entropy production in a one-dimensional system.

Let us consider a stochastic diffusive process that is driven away from equilibrium. More specifically, we have in mind a system where a gradient of density is imposed by appropriate boundary conditions. Let the system be essentially one-dimensional by imposing translational invariance in the other spatial directions. The relevant spatial direction is denoted by x, $0 < x < L+1$. At the boundaries, $x = 0$ and $x = L+1$, we put particle reservoirs with respective particle densities w_- and w_+. The reservoirs keep on feeding the system with new particles and those particles that exit never come back.

The probability density of a tracer particle is $w_t(x)$ and obeys the mass conservation law

$$\frac{\partial w_t(x)}{\partial t} + \vec{\nabla} \cdot \vec{j}_t(x) = 0, \tag{1}$$

where $\vec{j}_t(x)$ is the associated current.

To follow the phenomenological approach of non-equilibrium thermodynamics, we have to supplement the mass conservation law with a linear law that relates the current $\vec{j}_t(x)$ to the gradient of the density $\vec{\nabla} w_t(x)$,

$$\vec{j}_t(x) = -D\vec{\nabla} w_t(x), \tag{2}$$

where D is the diffusion coefficient associated with that process. This is known as Fick's law [3].

Combining Eqs. (1-2) and assuming that the diffusion coefficient is constant, we obtain the Fokker-Planck equation for diffusion,

$$\frac{\partial w_t(x)}{\partial t} = D\nabla^2 w_t(x). \tag{3}$$

The stationary solution of this equation is found by imposing the boundary conditions $w_t(0) = w_-$ and $w_t(L+1) = w_+$,

$$w(x) = w_- + (w_+ - w_-)\frac{x}{L+1}. \tag{4}$$

The connection with the second law of thermodynamics is made by considering the entropy whose local density is defined as

$$s_t(x) = -\{\log[w_t(x)] - 1\}, \tag{5}$$

and $S(t) = \int_0^{L+1} dx w_t(x) s_t(x)$ is the macroscopic entropy. Taking the derivative of the integrand with respect to time, we find successively

$$\begin{aligned}\frac{\partial w_t(x) s_t(x)}{\partial t} &= -\frac{\partial w_t(x)}{\partial t}\log[w_t(x)] \\ &= [\vec{\nabla} \cdot \vec{j}_t(x)]\log[w_t(x)] \\ &= \vec{\nabla} \cdot \{\vec{j}_t(x)\log[w_t(x)]\} + \frac{\vec{j}_t(x)^2}{D w_t(x)},\end{aligned} \tag{6}$$

where we used Eq. (1) in the second line and Eq. (2) in the third one. Equation (6) has the form of a local entropy balance [3]

$$\frac{\partial w_t(x) s_t(x)}{\partial t} = -\vec{\nabla} \cdot \vec{J}_s^{\mathrm{tot}}(x,t) + \sigma_t(x), \tag{7}$$

where $\vec{J}_s^{\mathrm{tot}}(x,t)$ is the total entropy flow and $\sigma_t(x) > 0$ is the entropy source term. Equation (7) can be rewritten in a slightly different form

$$w_t(x) \frac{ds_t(x)}{dt} = -\vec{\nabla} \cdot \vec{J}_s(x,t) + \sigma_t(x), \tag{8}$$

where the entropy flux is the difference

$$\vec{J}_s(x,t) = \vec{J}_s^{\mathrm{tot}}(x,t) - w_t(x) s_t(x) \vec{j}_t(x), \tag{9}$$

between the total entropy flux and a convective term.

Using Eq. (2) again, we find that the local rate of entropy production $\sigma_t(x)$ in Eq. (6) can be rewritten in terms of the density $w_t(x)$ only :

$$\sigma_t(x) = D \frac{[\vec{\nabla} w_t(x)]^2}{w_t(x)}. \tag{10}$$

In the stationary state, Eq. (4), the local rate of entropy production becomes

$$\sigma(x) = D \frac{(w_+ - w_-)^2}{(L+1)^2 w(x)}. \tag{11}$$

III. PERSISTENT RANDOM WALK – DEFINITIONS

We consider the following process : a particle on a one-dimensional lattice moves from site to site with a given velocity ± 1 and, at each time step, is scattered forward with probability p and backwards with probability q, such that $p + q = 1$. Thus a particle located at site $n \in \mathbf{Z}$ with velocity $v = \pm 1$ will move to $n \pm 1$ conserving or reversing its velocity with the probabilities :

$$(n, v) \quad \rightarrow \quad \begin{cases} (n+v, v) & \text{with probability } p, \\ (n-v, -v) & \text{with probability } q. \end{cases} \tag{12}$$

The diffusion coefficient for this process is [16,18,17]

$$D = \frac{p}{2q}. \tag{13}$$

To construct a deterministic model of this process, we start with a procedure similar to the one used by Dorfman, Ernst and Jacobs [5,4], and consider the velocity of the particle only and associate to it a unit interval. Let us divide the interval into two halves, corresponding to the two possible values of the velocity. For definiteness, let us assign the first half to $v = -1$ and the second to $v = +1$. According to Eq. (12), a particle has a probability p of keeping its velocity unchanged and q to reverse it. Therefore we have to further subdivide both halves into two parts of respective lengths $p/2$ and $q/2$, the first of which must be mapped onto the same half and the second onto the other half. Each of these branches must be linear and onto in order to model the independent process of velocity flips. This is illustrated in Fig. (1) for the case $q = 1/3$. Therefore a one-dimensional map whose dynamics mimics the random sequence of the velocities is given by

$$\phi_q(x) = \begin{cases} \frac{x}{q} + \frac{1}{2}, & 0 \le x < \frac{q}{2}, \\ \frac{x - \frac{q}{2}}{p}, & \frac{q}{2} \le x < 1 - \frac{q}{2}, \\ \frac{x - 1 + \frac{q}{2}}{q}, & 1 - \frac{q}{2} \le x < 1. \end{cases} \tag{14}$$

The generalization of $\Phi_q(x)$ to a multi-baker map is straightforward :

$$M_q^{\rm prw}(n,x,y) = \begin{cases} \left(n+1, \frac{x}{q}+\frac{1}{2}, q\left(y-\frac{1}{2}\right)+1\right), & 0 \le x < \frac{q}{2},\ 0 \le y < \frac{1}{2}, \\ \left(n+1, \frac{x}{q}+\frac{1}{2}, q\left(y-\frac{1}{2}\right)\right), & 0 \le x < \frac{q}{2},\ \frac{1}{2} \le y < 1, \\ \left(n-1, \frac{x-q/2}{p}, py+\frac{q}{2}\right), & \frac{q}{2} \le x < \frac{1}{2}, \\ \left(n+1, \frac{x-q/2}{p}, py+\frac{q}{2}\right), & \frac{1}{2} \le x < 1-\frac{q}{2}, \\ \left(n-1, \frac{x-1+q/2}{q}, q\left(y-\frac{1}{2}\right)+1\right), & 1-\frac{q}{2} \le x < 1,\ 0 \le y < \frac{1}{2}, \\ \left(n-1, \frac{x-1+q/2}{q}, q\left(y-\frac{1}{2}\right)\right), & 1-\frac{q}{2} \le x < 1,\ \frac{1}{2} \le y < 1. \end{cases} \tag{15}$$

This map is shown in Fig(2). We note that if one restricts this map to a single unit square, by ignoring the changes in the site index, n, this map is reversing under the time-reversal operators of the usual baker map $T(x,y) = (1-y, 1-x)$ and $S(x,y) = (y,x)$, i. e.

$$T \circ M_q^{\rm prw} \circ T(x,y) = M_q^{\rm prw\,-1}(x,y), \tag{16}$$

$$S \circ M_q^{\rm prw} \circ S(x,y) = M_q^{\rm prw\,-1}(x,y),. \tag{17}$$

However, the full multi-baker map, with the changes in the site index taken into account, reversing under either of these two time-reversal operators. Indeed, one finds that, depending on x and y, the composition

$$T \circ M_q^{\rm prw} \circ T \circ M_q^{\rm prw}(n,x,y) = \left(\left\{ \begin{array}{l} n+2 \\ n \\ n-2 \end{array} \right\}, x, y \right), \tag{18}$$

and similarly for S. This is displayed in Figs. (3-4). It is interesting to note that we can combine the action of S and T to form a hybrid operator U that is reversing. Indeed, as seen in Fig. (5), S and T act on the square in such a way that the images of the elements of the natural partition of the square under these two operators do not overlap. Therefore U, defined by

$$U(x,y) = \begin{cases} T(x,y), & 0 \le x < \frac{1}{2}, & \frac{q}{2} \le y < \frac{1}{2}, \\ & & 1-\frac{q}{2} \le y < 1, \\ & \frac{1}{2} \le x < 1, & 0 \le y < \frac{q}{2}, \\ & & \frac{1}{2} \le y < 1-\frac{q}{2}, \\ S(x,y) & \text{otherwise}, \end{cases} \tag{19}$$

is a reversing operator for $M_q^{\rm prw}$. However, contrary to T and S, U is not an involution, i.e. $U \circ U(x,y) \neq (x,y)$, and therefore U is not necessarily a reversing symmetry of any power of $M_q^{\rm prw}$. Actually $U \circ U$ is an involution,

$$U^4(x,y) = U \circ U \circ U \circ U(x,y) = (x,y). \tag{20}$$

This implies that the conjugation of U with odd powers of $M_q^{\rm prw}$ yields its inverse, while the conjugation with even powers leaves the map unchanged. In other words, U is a reversing symmetry of $M_q^{{\rm prw}\,n}$ for n odd but a symmetry for n even :

$$U \circ M_q^{{\rm prw}\,n} \circ U = \begin{cases} M_q^{{\rm prw}\,-n}, & n \text{ odd}, \\ M_q^{{\rm prw}\,n}, & n \text{ even}, \end{cases} \tag{21}$$

This property of a reversing operator can be contrasted with the existence of two reversing symmetries for the maps on the unit square, which itself implies the existence of a non-trivial symmetry, i.e. the composition of T and S. Those properties are well established [11], while the property described by Eq. (21) is new, to our knowledge.

The weakened reversibility of the map $M_q^{\rm prw}$, Eq. (21), still allows the map to have a properly behaved dynamical entropy production as we will see in Sec. VI.

IV. STATIONARY STATE UNDER FLUX BOUNDARY CONDITIONS

As with the open multi-baker map [14,7,8], we impose flux boundary conditions. We consider a chain of L sites and study the distribution of an infinite number of copies of identical systems imposing that, in average, 1 particle is present at the left-end of the chain, regardless of its velocity and $L+2$ particles at the right-end.

The stationary measure is found by considering the following cumulative functions :

$$G^{(-)}(n,x,y) = \int_0^x dx' \int_0^y dy' \rho(n,x',y'), \quad 0 \le x < \frac{1}{2}, \tag{22}$$

$$G^{(+)}(n,x,y) = \int_{1/2}^x dx' \int_0^y dy' \rho(n,x',y'), \quad \frac{1}{2} \le x < 1, \tag{23}$$

where $\rho(n,x,y)$ is the corresponding density function. Because the map is piecewise uniformly expanding along the x direction, the invariant measure is uniform along the x-direction and we therefore have

$$G^{(-)}(n,x,y) = 2x g^{(-)}(n,y), \quad G^{(+)}(n,x,y) = (2x-1) g^{(+)}(n,y), \tag{24}$$

where the functions $g^{(-)}$ and $g^{(+)}$ are solutions of the following set of equations :

$$g^{(\mp)}(n,y) = \begin{cases} q g^{(\pm)}\left(n \pm 1, \frac{y}{q} + \frac{1}{2}\right) - q g^{(\pm)}\left(n \pm 1, \frac{1}{2}\right), \quad 0 \le y < \frac{q}{2}, \\ q g^{(\pm)}(n \pm 1, 1) - q g^{(\pm)}\left(n \pm 1, \frac{1}{2}\right) + p g^{(\mp)}\left(n \pm 1, \dfrac{y - q/2}{p}\right), \\ \qquad \frac{q}{2} \le y < 1 - \frac{q}{2}, \\ q g^{(\pm)}(n \pm 1, 1) - q g^{(\pm)}\left(n \pm 1, \frac{1}{2}\right) + p g^{(\mp)}(n \pm 1, 1) \\ \quad + q g^{(\pm)}\left(n \pm 1, \dfrac{y-1}{q} + \frac{1}{2}\right), \quad 1 - \frac{q}{2} \le y < 1, \end{cases} \tag{25}$$

With flux boundary conditions,

$$g^{(-)}(0,y) + g^{(+)}(0,y) = y, \quad g^{(-)}(L+1,y) + g^{(+)}(L+1,y) = (L+2)y, \tag{26}$$

the solutions of (25) for $y=1$ are readily found to be

$$g^{(\pm)}(n,1) = \frac{n+1}{2} \mp \frac{1}{4q}. \tag{27}$$

So that, for arbitrary y, the solution takes the form

$$g^{(\pm)}(n,y) = \left(\frac{n+1}{2} \mp \frac{1}{4q}\right) y \pm T_n^{(\pm)}(y), \tag{28}$$

where the class of functions $T_n^{(\pm)}$ is a one-parameter family, i.e. implicitly dependent on the scattering parameter q, of generalized incomplete Takagi functions,

$$T_n^{(\pm)}(y) = \begin{cases} \frac{p}{2q} y - q T_{n\mp1}^{(\mp)}\left(\frac{y}{q} + \frac{1}{2}\right), & 0 \le y < \frac{q}{2}, \\ \dfrac{1/2 - y}{2} + p T_{n\mp1}^{(\pm)}\left(\dfrac{y - q/2}{p}\right), & \frac{q}{2} \le y < 1 - \frac{q}{2}, \\ -\frac{p}{2q}(1-y) - q T_{n\mp1}^{(\mp)}\left(\dfrac{y-1}{q} + \frac{1}{2}\right), & 1 - \frac{q}{2} \le y < 1, \end{cases} \tag{29}$$

where $1 \le n \le L$ and the boundary conditions on the functions $T_n^{(\pm)}(y)$ are

$$T_0^{(+)}(y) = 0, \quad T_{L+1}^{(-)}(y) = 0. \tag{30}$$

In Fig. (6), we display the functions $T_n^{(\pm)}$ for the sites $n = 1,3,5$ on a chain of $L = 100$ sites and the value of the scattering parameter $q = 1/3$.

We now want to simplify the argument and will make the assumption in the sequel that we are far enough from the boundaries so that we can ignore the finite size effects and replace the incomplete functions $T_n^{(\pm)}$ by their common limit value

$$
T_q(y) = \begin{cases} \frac{p}{2q} y - q T_q\left(\frac{y}{q} + \frac{1}{2}\right), & 0 \le y < \frac{q}{2}, \\ \frac{1/2 - y}{2} + p T_q\left(\frac{y - q/2}{p}\right), & \frac{q}{2} \le y < 1 - \frac{q}{2}, \\ -\frac{p}{2q}(1-y) - q T_q\left(\frac{y-1}{q} + \frac{1}{2}\right), & 1 - \frac{q}{2} \le y < 1, \end{cases} \tag{31}
$$

where we have made explicit the parametric dependence of the generalized Takagi functions T_q. Figure (7) shows a recursive computation of $T_q(y)$, for $q = 1/3$. This example is very similar to the Takagi function [13]

$$
T(y) = \begin{cases} y + \frac{1}{2} T(2y), & 0 \le y < \frac{1}{2}, \\ 1 - y + \frac{1}{2} T(2y - 1), & \frac{1}{2} \le y < 1, \end{cases} \tag{32}
$$

that appears in the example of the open random walk with flux boundary conditions discussed by Gaspard and co-workers [14,7,8]. In particular, for $q = 1/2$, the case of a symmetric persistent random walk, up to a factor 2, T_q is identical to T on the first half of the interval and opposite on the second half. Figure (8) shows $T_q(y)$ for different values of q ranging from 0.1 to 0.9.

V. SYMBOLIC DYNAMICS

In order to compute the entropy production, it is convenient to introduce a symbolic dynamics. Each half of the unit cell is partitioned into sets that correspond to particles that were back- or forward-scattered at the preceding time step. This defines the 0-partition of a cell,

$$
\mathcal{A} = \{\Gamma_0^{(-)}, \Gamma_0^{(+)}, \Gamma_1^{(-)}, \Gamma_1^{(+)}\}, \tag{33}
$$

where

$$
\Gamma_i^{(\pm)} = \Gamma^{(\pm)} \cap \Gamma_i, \; i = 0, 1, \tag{34}
$$

and

$$
\Gamma^{(-)} = \left\{(x, y) \;:\; 0 \le x < \frac{1}{2}\right\}, \tag{35}
$$

$$
\Gamma^{(+)} = \left\{(x, y) \;:\; \frac{1}{2} \le x < 1\right\}, \tag{36}
$$

$$
\Gamma_0 = \left\{(x, y) \;:\; 0 \le y < \frac{q}{2} \text{ or } 1 - \frac{q}{2} \le y < 1\right\}, \tag{37}
$$

$$
\Gamma_1 = \left\{(x, y) \;:\; \frac{q}{2} \le y < 1 - \frac{q}{2}\right\}. \tag{38}
$$

An (l, k)-partition is the collection of cylinder sets

$$
\Gamma^{(\pm)}_{\omega_{-l},\ldots,\omega_k} = \Gamma^{(\pm)} \cap \left[\cap_{i=-l}^{k} M_q^{\mathrm{prw}\, i}(\Gamma_{\omega_i})\right], \tag{39}
$$

where $\omega_i \in \{0, 1\}$, $i = -l, \ldots, k$. The measure of such a set, $\mu_n^{(\pm)}(\omega_{-l}, \ldots, \omega_k)$, can be written, for $l = 0$, as

$$
\Delta g_n^{(\pm)}(\omega_0, \ldots, \omega_k) = g^{(\pm)}\Big(n, y(\omega_0, \ldots, \omega_k + 1)\Big) - g^{(\pm)}\Big(n, y(\omega_0, \ldots, \omega_k)\Big), \tag{40}
$$

where $\omega_0, \ldots, \omega_{k-1}, \omega_k + 1$ is defined by

$$
\omega_0, \ldots, \omega_{k-1}, \omega_k + 1 = \begin{cases} \omega_0, \ldots, \omega_{k-1}, 1, & \omega_k = 0, \\ \omega_0, \ldots, \omega_{k-1} + 1, 0 & \omega_k = 1, \end{cases} \tag{41}
$$

with the further convention that $y(1, \ldots, 1, 1 + 1) = 1$.

By looking at the y-components of M_q^{prw} in Eq. (15), we find that $y(\omega_0, \ldots, \omega_k)$ can be computed through the recursion relation

$$y(\omega_0,\dots,\omega_k) = \Big(\nu(\omega_0) y(\omega_1,\dots,\omega_k) - (-)^{\omega_0}\frac{q}{2}\Big) \bmod 1, \tag{42}$$

where we set

$$\nu(\omega_0) = \begin{cases} q, & \omega_0 = 0,\\ p, & \omega_0 = 1. \end{cases} \tag{43}$$

Therefore the functions T_q in Eq. (31) can be defined directly in terms of the symbolic sequences,

$$T_q(\omega_0,\dots,\omega_k) = \begin{cases} \frac{p}{2}\Big(y(\omega_1,\dots,\omega_k) - \frac{1}{2}\Big) - qT_q(\omega_1,\dots,\omega_k), & \omega_0 = 0,\\ -\frac{p}{2}\Big(y(\omega_1,\dots,\omega_k) - \frac{1}{2}\Big) + pT_q(\omega_1,\dots,\omega_k), & \omega_0 = 1. \end{cases} \tag{44}$$

With the help of Eq. (28), we can then rewrite Eq. (40) as

$$\Delta g_n^{(\pm)}(\omega_0,\dots,\omega_k) = \left(\frac{n+1}{2} \mp \frac{1}{4q}\right)\Delta y(\omega_0,\dots,\omega_k) \pm \Delta T_q(\omega_0,\dots,\omega_k), \tag{45}$$

with obvious notations for Δy and ΔT_q. We note here that our assumption that the cell n is sufficiently far away from the boundaries means in terms of the size of the cylinder sets that k must be strictly less than the distance to the closest boundary. In that case, we can substitute without loss of generality $T_n^{(\pm)}$ by T_q in Eq. (28).

For later purposes, we note that, from Eqs. (42, 44), we have

$$\Delta y(\omega_0,\dots,\omega_k) = \nu(\omega_0)\Delta y(\omega_1,\dots,\omega_k) = \prod_{i=0}^{k} \nu(\omega_i), \tag{46}$$

and

$$\Delta T_q(\omega_0,\dots,\omega_k) = \begin{cases} \frac{p}{2}\Delta y(\omega_1,\dots,\omega_k) - q\Delta T_q(\omega_1,\dots,\omega_k), & \omega_0 = 0,\\ -\frac{p}{2}\Delta y(\omega_1,\dots,\omega_k) + p\Delta T_q(\omega_1,\dots,\omega_k), & \omega_0 = 1. \end{cases} \tag{47}$$

VI. k-ENTROPY AND ENTROPY PRODUCTION RATE

Following [9], we write the entropy of a (l,k)-partition as

$$S_{l,k}(n) \equiv S_{l,k}^{(-)}(n) + S_{l,k}^{(+)}(n), \tag{48}$$

and

$$S_{l,k}^{(\pm)}(n) = -\sum_{\omega_{-l},\dots,\omega_{k-1}} \mu_n^{(\pm)}(\omega_{-l},\dots,\omega_{k-1})\left[\log\frac{\mu_n^{(\pm)}(\omega_{-l},\dots,\omega_{k-1})}{\nu^{(\pm)}(\omega_{-l},\dots,\omega_{k-1})} - 1\right], \tag{49}$$

Because the invariant measure is uniform along the expanding direction, the ratio of $\mu_n^{(\pm)}$ and $\nu^{(\pm)}$ obeys the identity

$$\frac{\mu_n^{(\pm)}(\omega_{-l},\dots,\omega_{k-1})}{\nu^{(\pm)}(\omega_{-l},\dots,\omega_{k-1})} = \frac{\mu_n^{(\pm)}(\omega_0,\dots,\omega_{k-1})}{\nu^{(\pm)}(\omega_0,\dots,\omega_{k-1})}. \tag{50}$$

Therefore the entropy is extensive with respect to the x-direction and we have

$$S_{l,k}^{(\pm)}(n) = S_{0,k}^{(\pm)}(n). \tag{51}$$

Henceforth, we will drop the l dependence and will simply consider the k-entropies, which we write

$$S_k^{(\pm)}(n) = -\sum_{\underline{\omega}_k} \Delta g_n^{(\pm)}(\underline{\omega}_k)\left[\log\frac{\Delta g_n^{(\pm)}(\underline{\omega}_k)}{\nu^{(\pm)}(\underline{\omega}_k)} - 1\right], \tag{52}$$

where we used the compact notation $\underline{\omega}_k \equiv \omega_0, \ldots, \omega_{k-1}$. Here $\nu^{(\pm)}(\underline{\omega}_k)$ is the volume of the corresponding cylinder set,

$$\nu^{(\pm)}(\underline{\omega}_k) = \frac{1}{2}\Delta y(\underline{\omega}_k) = \frac{1}{2}\prod_{i=o}^{k-1}\nu(\omega_i). \tag{53}$$

The entropy production rate follows by a straightforward generalization of the formalism detailed in [9] to Eq. (52) :

$$\Delta_i S_k(n) \equiv \Delta_i S_k^{(-)}(n) + \Delta_i S_k^{(+)}(n), \tag{54}$$

and

$$\Delta_i S_k^{(\pm)}(n) = \sum_{\underline{\omega}_{k+1}} \Delta g_n^{(\pm)}(\underline{\omega}_{k+1}) \log \frac{\Delta g_n^{(\pm)}(\underline{\omega}_{k+1})}{\nu(\omega_k)\Delta g_n^{(\pm)}(\underline{\omega}_k)}. \tag{55}$$

Assuming that the stationary state is dominated by the linear part, we can compute the k-entropy production rate by expanding the expressions for $\Delta_i S_k^{(\pm)}(n)$ in Eq. (55) in powers of

$$\frac{\Delta T_q}{(\frac{n+1}{2} \pm \frac{1}{4q})\Delta y}, \tag{56}$$

which we further expand in powers of $\frac{1}{n+1}$. To first order, Eq. (54) becomes

$$\Delta_i S_k(n) = \frac{2}{n+1}\sum_{\underline{\omega}_{k+1}} \frac{[\Delta T_q(\underline{\omega}_{k+1}) - \nu(\omega_k)\Delta T_q(\underline{\omega}_k)]^2}{\Delta y(\underline{\omega}_{k+1})}. \tag{57}$$

Making use of Eqs. (46-47), we readily see that this expression is independent of k, i.e.

$$\begin{aligned}
&\sum_{\underline{\omega}_{k+1}} \frac{[\Delta T_q(\underline{\omega}_{k+1}) - \nu(\omega_k)\Delta T_q(\underline{\omega}_k)]^2}{\Delta y(\underline{\omega}_{k+1})} \\
&= \sum_{\underline{\omega}_k} \frac{[\Delta T_q(\underline{\omega}_k) - \nu(\omega_{k-1})\Delta T_q(\underline{\omega}_{k-1})]^2}{\Delta y(\underline{\omega}_k)} \\
&= \sum_{\omega=0,1} \frac{[\Delta T_q(\omega)]^2}{\Delta y(\omega)} \\
&= \frac{p}{4q}.
\end{aligned} \tag{58}$$

Therefore,

$$\Delta_i S_k(n) = \frac{p}{2q}\frac{1}{n+1} + O\left(\frac{1}{(n+1)^3}\right). \tag{59}$$

The derivation of the next order term proceeds as follows. For the sake of simplifying the notations, we will write Eq. (27) as

$$g_n^{(\pm)} \equiv g^{(\pm)}(n, 1) \tag{60}$$

We start from Eq. (55) and substitute Eq. (45) for $\Delta g_n^{(\pm)}(\underline{\omega}_{k+1})$:

$$\begin{aligned}
\Delta_i S_k^{(\pm)}(n) &= \sum_{\underline{\omega}_{k+1}} \Delta g_n^{(\pm)}(\underline{\omega}_k + 1) \log \frac{\Delta g_n^{(\pm)}(\underline{\omega}_{k+1})}{\nu(\omega_k)\Delta g_n^{(\pm)}(\underline{\omega}_k)}, \\
&= \sum_{\underline{\omega}_{k+1}} \left[g_n^{(\pm)}\Delta y(\underline{\omega}_{k+1}) \pm \Delta T_q(\underline{\omega}_{k+1})\right] \\
&\qquad \log \frac{\left[g_n^{(\pm)}\Delta y(\underline{\omega}_{k+1}) \pm \Delta T_q(\underline{\omega}_{k+1})\right]}{\left[g_n^{(\pm)}\Delta y(\underline{\omega}_{k+1}) \pm \nu(\omega_k)\Delta T_q(\underline{\omega}_k)\right]},
\end{aligned} \tag{61}$$

where, in the last line, we used

$$\nu(\omega_k)\Delta y(\underline{\omega}_k) = \Delta y(\underline{\omega}_{k+1}). \tag{62}$$

Factoring $g_n^{(\pm)}\Delta y(\underline{\omega}_{k+1})$ in the logarithms and expanding up to fourth order, Eq.(61) becomes

$$\begin{aligned}
&\sum_{\underline{\omega}_{k+1}} \left[g_n^{(\pm)}\Delta y(\underline{\omega}_{k+1}) \pm \Delta T_q(\underline{\omega}_{k+1})\right] \\
&\qquad \left\{\log\left[1 \pm \frac{\Delta T_q(\underline{\omega}_{k+1})}{g_n^{(\pm)}\Delta y(\underline{\omega}_{k+1})}\right] - \log\left[1 \pm \frac{\nu(\omega_k)\Delta T_q(\underline{\omega}_k)}{g_n^{(\pm)}\Delta y(\underline{\omega}_{k+1})}\right]\right\} = \\
&\sum_{\underline{\omega}_{k+1}} \left[g_n^{(\pm)}\Delta y(\underline{\omega}_{k+1}) \pm \Delta T_q(\underline{\omega}_{k+1})\right] \left\{ \pm \frac{\Delta T_q(\underline{\omega}_{k+1})}{g_n^{(\pm)}\Delta y(\underline{\omega}_{k+1})}\right. \\
&\mp \frac{\nu(\omega_k)\Delta T_q(\underline{\omega}_k)}{g_n^{(\pm)}\Delta y(\underline{\omega}_{k+1})} - \frac{\Delta T_q(\underline{\omega}_{k+1})^2}{2\left[g_n^{(\pm)}\Delta y(\underline{\omega}_{k+1})\right]^2} \\
&+ \frac{\nu(\omega_k)^2\Delta T_q(\underline{\omega}_k)^2}{2\left[g_n^{(\pm)}\Delta y(\underline{\omega}_{k+1})\right]^2} \pm \frac{\Delta T_q(\underline{\omega}_{k+1})^3}{3\left[g_n^{(\pm)}\Delta y(\underline{\omega}_{k+1})\right]^3} \mp \frac{\nu(\omega_k)^3\Delta T_q(\underline{\omega}_k)^3}{3\left[g_n^{(\pm)}\Delta y(\underline{\omega}_{k+1})\right]^3} \\
&\left. - \frac{\Delta T_q(\underline{\omega}_{k+1})^4}{4\left[g_n^{(\pm)}\Delta y(\underline{\omega}_{k+1})\right]^4} + \frac{\nu(\omega_k)^4\Delta T_q(\underline{\omega}_k)^4}{4\left[g_n^{(\pm)}\Delta y(\underline{\omega}_{k+1})\right]^4}\right\},
\end{aligned} \tag{63}$$

which, keeping terms up to $O(1/(n+1)^3)$, takes the form

$$\begin{aligned}
&\sum_{\underline{\omega}_{k+1}} \left\{ \pm \Delta T_q(\underline{\omega}_{k+1}) \mp \nu(\omega_k)\Delta T_q(\underline{\omega}_k) + \frac{\Delta T_q(\underline{\omega}_{k+1})^2}{2g_n^{(\pm)}\Delta y(\underline{\omega}_{k+1})}\right. \\
&- \frac{\nu(\omega_k)\Delta T_q(\underline{\omega}_{k+1})\Delta T_q(\underline{\omega}_k)}{g_n^{(\pm)}\Delta y(\underline{\omega}_{k+1})} + \frac{\nu(\omega_k)^2\Delta T_q(\underline{\omega}_k)^2}{2g_n^{(\pm)}\Delta y(\underline{\omega}_{k+1})} \\
&\mp \frac{\Delta T_q(\underline{\omega}_{k+1})^3}{6\left[g_n^{(\pm)}\Delta y(\underline{\omega}_{k+1})\right]^2} \pm \frac{\nu(\omega_k)^2\Delta T_q(\underline{\omega}_{k+1})\Delta T_q(\underline{\omega}_k)^2}{2\left[g_n^{(\pm)}\Delta y(\underline{\omega}_{k+1})\right]^2} \\
&\mp \frac{\nu(\omega_k)^3\Delta T_q(\underline{\omega}_k)^3}{3\left[g_n^{(\pm)}\Delta y(\underline{\omega}_{k+1})\right]^2} + \frac{\Delta T_q(\underline{\omega}_{k+1})^4}{12\left[g_n^{(\pm)}\Delta y(\underline{\omega}_{k+1})\right]^3} \\
&\left. - \frac{\nu(\omega_k)^3\Delta T_q(\underline{\omega}_{k+1})\Delta T_q(\underline{\omega}_k)^3}{3\left[g_n^{(\pm)}\Delta y(\underline{\omega}_{k+1})\right]^3} + \frac{\nu(\omega_k)^4\Delta T_q(\underline{\omega}_k)^4}{4\left[g_n^{(\pm)}\Delta y(\underline{\omega}_{k+1})\right]^3}\right\}.
\end{aligned} \tag{64}$$

Therefore, the k-entropy production rate, Eq.(54), reads

$$\begin{aligned}
\Delta_i S_k(n) = &\left(\frac{1}{2g_n^{(+)}} + \frac{1}{2g_n^{(-)}}\right) \sum_{\underline{\omega}_{k+1}} \frac{\left[\Delta T_q(\underline{\omega}_{k+1}) - \nu(\omega_k)\Delta T_q(\underline{\omega}_k)\right]^2}{\Delta y(\underline{\omega}_{k+1})} \\
&+ \left(\frac{1}{6{g_n^{(+)}}^2} - \frac{1}{6{g_n^{(-)}}^2}\right) \\
&\sum_{\underline{\omega}_{k+1}} \frac{-\Delta T_q(\underline{\omega}_{k+1})^3 + 3\nu(\omega_k)^2\Delta T_q(\underline{\omega}_{k+1})\Delta T_q(\underline{\omega}_k)^2 - 2\nu(\omega_k)^3\Delta T_q(\underline{\omega}_k)^3}{\Delta y(\underline{\omega}_{k+1})^2} \\
&+ \left(\frac{1}{12{g_n^{(+)}}^3} + \frac{1}{12{g_n^{(-)}}^3}\right)
\end{aligned}$$

$$\sum_{\underline{\omega}_{k+1}} \frac{\Delta T_q(\underline{\omega}_{k+1})^4 - 4\nu(\omega_k)^3 \Delta T_q(\underline{\omega}_{k+1}) \Delta T_q(\underline{\omega}_k)^3 + 3\nu(\omega_k)^4 \Delta T_q(\underline{\omega}_k)^4}{\Delta y(\underline{\omega}_{k+1})^3}. \tag{65}$$

Assuming that n is large, we can expand the ratios involving $g_n^{(\pm)}$ in powers of $1/(n+1)$. To third order, we get

$$\frac{1}{2g_n^{(+)}} + \frac{1}{2g_n^{(-)}} = \frac{2}{n+1} + \frac{1}{2q^2(n+1)^3}, \tag{66}$$

$$\frac{1}{6{g_n^{(+)}}^2} - \frac{1}{6{g_n^{(-)}}^2} = \frac{4}{3q(n+1)^3}, \tag{67}$$

$$\frac{1}{12{g_n^{(+)}}^3} + \frac{1}{12{g_n^{(-)}}^3} = \frac{4}{3(n+1)^3}. \tag{68}$$

Therefore, to leading order in $1/(n+1)$, we retrieve Eq. (57), which, along with Eq. (58), gives Eq.(59).

To work out the next order terms, we first note, with the help of Eq.(62), that expressions involving both $\Delta T_q(\underline{\omega}_{k+1})$ and $\Delta T_q(\underline{\omega}_k)$ can be transformed into expressions involving only $\Delta T_q(\underline{\omega}_k)$ by summing over ω_k. For instance,

$$\frac{\nu(\omega_k)^2 \Delta T_q(\underline{\omega}_{k+1}) \Delta T_q(\underline{\omega}_k)^2}{\Delta y(\underline{\omega}_{k+1})^2} = \frac{\Delta T_q(\underline{\omega}_{k+1}) \Delta T_q(\underline{\omega}_k)^2}{\Delta y(\underline{\omega}_k)^2}. \tag{69}$$

Now, by definition,

$$\begin{aligned}
\sum_{\omega_k} \Delta T_q(\underline{\omega}_{k+1}) &= \sum_{\omega_k} \left[T_q(\underline{\omega}_k, \omega_k + 1) - T_q(\underline{\omega}_k, \omega_k) \right], \\
&= T_q(\underline{\omega}_{k-1}, \omega_{k-1} + 1) - T_q(\underline{\omega}_k), \\
&= \Delta T_q(\underline{\omega}_k).
\end{aligned} \tag{70}$$

Therefore,

$$\sum_{\omega_k} \frac{\nu(\omega_k)^2 \Delta T_q(\underline{\omega}_{k+1}) \Delta T_q(\underline{\omega}_k)^2}{\Delta y(\underline{\omega}_{k+1})^2} = \frac{\Delta T_q(\underline{\omega}_k)^3}{\Delta y(\underline{\omega}_k)^2}, \tag{71}$$

and, similarly,

$$\sum_{\omega_k} \frac{\nu(\omega_k)^3 \Delta T_q(\underline{\omega}_{k+1}) \Delta T_q(\underline{\omega}_k)^3}{\Delta y(\underline{\omega}_{k+1})^3} = \frac{\Delta T_q(\underline{\omega}_k)^4}{\Delta y(\underline{\omega}_k)^3}. \tag{72}$$

Combining Eqs. (66-67, 69-72), we can rewrite the last two lines of Eq. (65) as

$$\begin{aligned}
&-\frac{4}{3q(n+1)^3} \left[\sum_{\underline{\omega}_{k+1}} \frac{\Delta T_q(\underline{\omega}_{k+1})^3}{\Delta y(\underline{\omega}_{k+1})^2} - \sum_{\underline{\omega}_k} \frac{\Delta T_q(\underline{\omega}_k)^3}{\Delta y(\underline{\omega}_k)^2} \right] \\
&+\frac{4}{3(n+1)^3} \left[\sum_{\underline{\omega}_{k+1}} \frac{\Delta T_q(\underline{\omega}_{k+1})^4}{\Delta y(\underline{\omega}_{k+1})^3} - \sum_{\underline{\omega}_k} \frac{\Delta T_q(\underline{\omega}_k)^4}{\Delta y(\underline{\omega}_k)^3} \right].
\end{aligned} \tag{73}$$

We can work out these expressions by substituting Eq.(47) for $\Delta T_q(\underline{\omega}_{k+1})$ and summing over ω_0. Before doing so, we will need the following

$$\sum_{\underline{\omega}_{k+1}} \Delta y(\underline{\omega}_{k+1}) = 1, \tag{74}$$

$$\sum_{\underline{\omega}_{k+1}} \Delta T_q(\underline{\omega}_{k+1}) = \sum_{\omega} \Delta T_q(\omega) = 0, \tag{75}$$

$$\sum_{\underline{\omega}_{k+1}} \frac{\Delta T_q(\underline{\omega}_{k+1})^2}{\Delta y(\underline{\omega}_{k+1})} = \frac{\left[\frac{p}{2}\Delta y(\underline{\omega}_k) - q\Delta T_q(\underline{\omega}_k)\right]^2}{q\Delta y(\underline{\omega}_k)}$$
$$+\frac{\left[\frac{p}{2}\Delta y(\underline{\omega}_k) - p\Delta T_q(\underline{\omega}_k)\right]^2}{p\Delta y(\underline{\omega}_k)},$$
$$= \frac{p^2}{4}\left(\frac{1}{q} + \frac{1}{p}\right) + \sum_{\underline{\omega}_k} \frac{\Delta T_q(\underline{\omega}_k)^2}{\Delta y(\underline{\omega}_k)},$$
$$= (k+1)\frac{p}{4q}. \tag{76}$$

For the cubic term, we have

$$\sum_{\underline{\omega}_{k+1}} \frac{\Delta T_q(\underline{\omega}_{k+1})^3}{\Delta y(\underline{\omega}_{k+1})^2} = \sum_{\underline{\omega}_k} \frac{\left[\frac{p}{2}\Delta y(\underline{\omega}_k) - q\Delta T_q(\underline{\omega}_k)\right]^3}{q^2\Delta y(\underline{\omega}_k)^2}$$
$$-\frac{\left[\frac{p}{2}\Delta y(\underline{\omega}_k) - p\Delta T_q(\underline{\omega}_k)\right]^3}{p^2\Delta y(\underline{\omega}_k)^2},$$
$$= \frac{p^3}{8}\left(\frac{1}{q^2} - \frac{1}{p^2}\right) + (p-q)\sum_{\underline{\omega}_k} \frac{\Delta T_q(\underline{\omega}_k)^3}{\Delta y(\underline{\omega}_k)^2},$$
$$= \frac{p}{8q^2}\sum_{i=1}^{k}(p-q)^i = \frac{p}{16q^3}(p-q)\left[1-(p-q)^k\right], \tag{77}$$

and, for the quartic term,

$$\sum_{\underline{\omega}_{k+1}} \frac{\Delta T_q(\underline{\omega}_{k+1})^4}{\Delta y(\underline{\omega}_{k+1})^3} = \sum_{\underline{\omega}_k} \frac{\left[\frac{p}{2}\Delta y(\underline{\omega}_k) - q\Delta T_q(\underline{\omega}_k)\right]^4}{q^3\Delta y(\underline{\omega}_k)^3}$$
$$+\frac{\left[\frac{p}{2}\Delta y(\underline{\omega}_k) - p\Delta T_q(\underline{\omega}_k)\right]^4}{p^3\Delta y(\underline{\omega}_k)^3},$$
$$= \frac{p^4}{16}\left(\frac{1}{q^3} + \frac{1}{p^3}\right) + k\frac{3p^2}{8q^2} - \frac{p^2}{4q^3}(p-q)\left[1-(p-q)^{k-1}\right]$$
$$+\sum_{\underline{\omega}_k} \frac{\Delta T_q(\underline{\omega}_k)^4}{\Delta y(\underline{\omega}_k)^3}. \tag{78}$$

Equation (65) combined with Eqs. (65, 66-68, 73, 77-78) yield the third order correction to the entropy production rate :

$$\Delta_i S_k(n) = \frac{p}{2q}\frac{1}{n+1} + \left[\frac{p}{q^3}\left(\frac{1}{8} - \frac{p^2}{3} + \frac{p^3}{12}\right)\right.$$
$$\left. + \frac{p}{12} + \frac{p^2}{3q^2} + \frac{p^2}{2q^2}k + (p-q)^{k+1}\frac{p}{6q^3}\right]\frac{1}{(n+1)^3}$$
$$+O\left(\frac{1}{(n+1)^5}\right). \tag{79}$$

We note that, in the symmetric case, $p = q = 1/2$, the coefficient in front of the third order term is $7/12 + k/2$. The term linear in k is identical to the case of the random walk [7,8], but the first term is different. For $q \neq p$, we find a new term, proportional to $(p-q)^k$, which decays exponentially. Thus, for large k, Eq. (79) is at least qualitatively similar to the case of the multi-baker map with a linear divergence in k that is third order in the small parameter $1/(n+1)$.

VII. DISCUSSION

We have shown that the entropy production formalism of Gaspard, Gilbert and Dorfman applies successfully to a persistent random walk driven away from equilibrium by a density current. The leading order term in the expansion in inverse powers of the spatial coordinate is in exact agreement with the phenomenological entropy production, Eq. (11). It is particularly important to note that this term is independent of the coarse-graining parameter, k. Our result is therefore similar to Gaspard's for the simpler case of a random walk [7,8], and we also find

$$\lim_{k\to\infty} \lim_{|\vec{\nabla}\mu_n|/\mu_n\to 0} \lim_{L\to\infty} \frac{\mu_n}{(\vec{\nabla}\mu_n)^2}\Delta_i S_k = D, \tag{80}$$

where we wrote $\mu_n = g(n,1)$ and $\vec{\nabla}$ denotes the density gradient with respect to the lattice coordinate. The limit on the resolution parameter is taken last, which is to say that the resolution dependent terms are accounted for by finite size effects and have thus no counterpart in thermodynamics, where one assumes that systems have infinitely many degrees of freedom.

It is interesting to note that the form of the first order contribution to the k-entropy production rate, Eq. (57), is rather universal. Indeed similar expressions arise in other baker map models [7,9,15]. It would therefore be interesting to understand in a more general setting the connection between the form of the generalized Takagi functions and the expression of the entropy production involving the diffusion coefficient. This question seems to be at the heart of the agreement between the dynamical and phenomenological approaches to entropy production and still needs further explanation.

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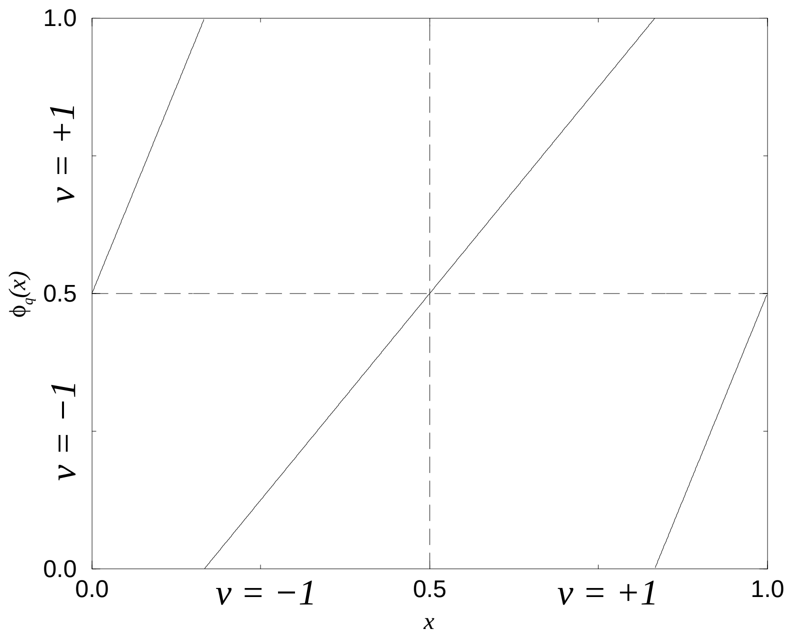

FIG. 1. The map ϕ_q defined by Eq. (14) mimics the probability rules, Eq. (12), for the velocity vector v.

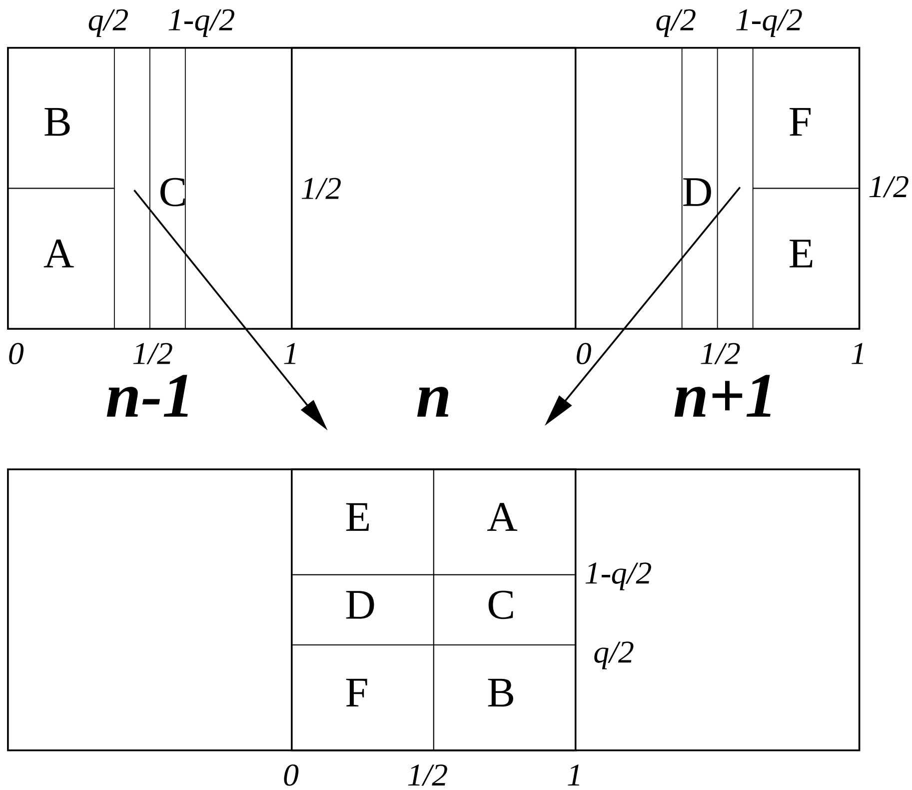

FIG. 2. The multi-baker map, Eq. (15), that models the persistent random walk, Eq. (12).

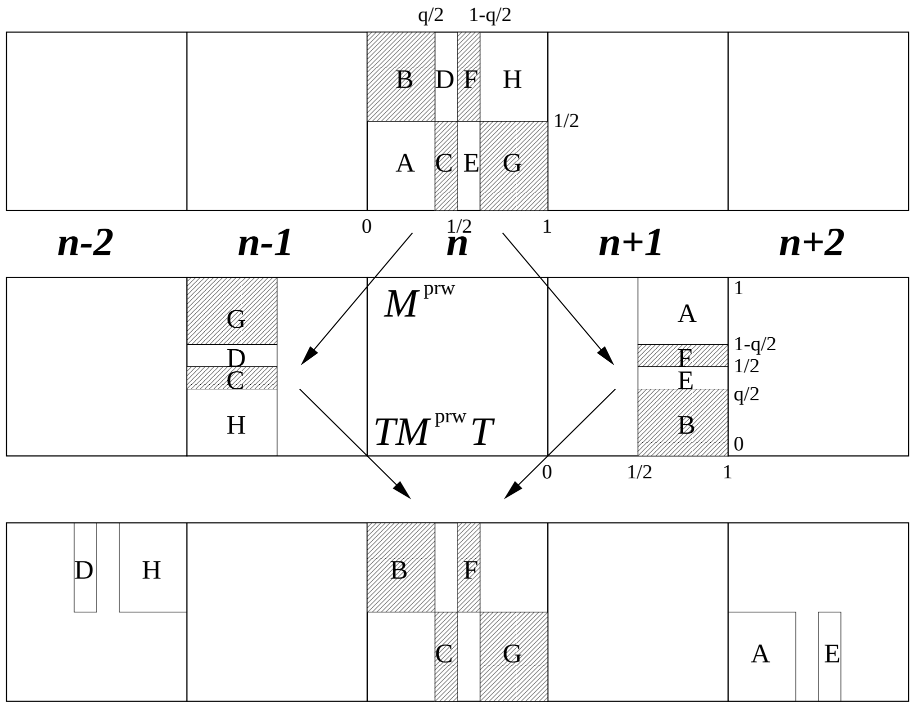

FIG. 3. The composition of M_q^{prw} with T, as in Eq. (18).

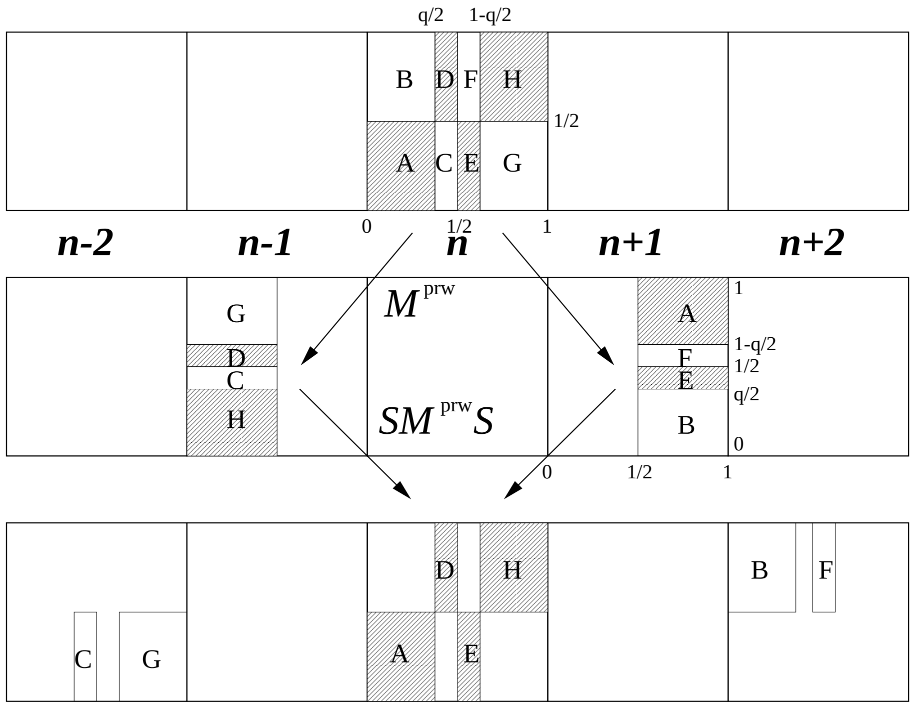

FIG. 4. The composition of M_q^{prw} with S, similar to Eq. (18).

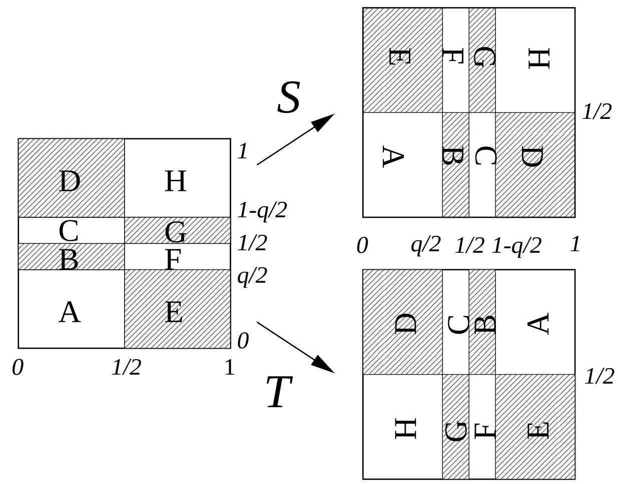

FIG. 5. The action of S and T on the elements of the partition induced by M_q^{prw}.

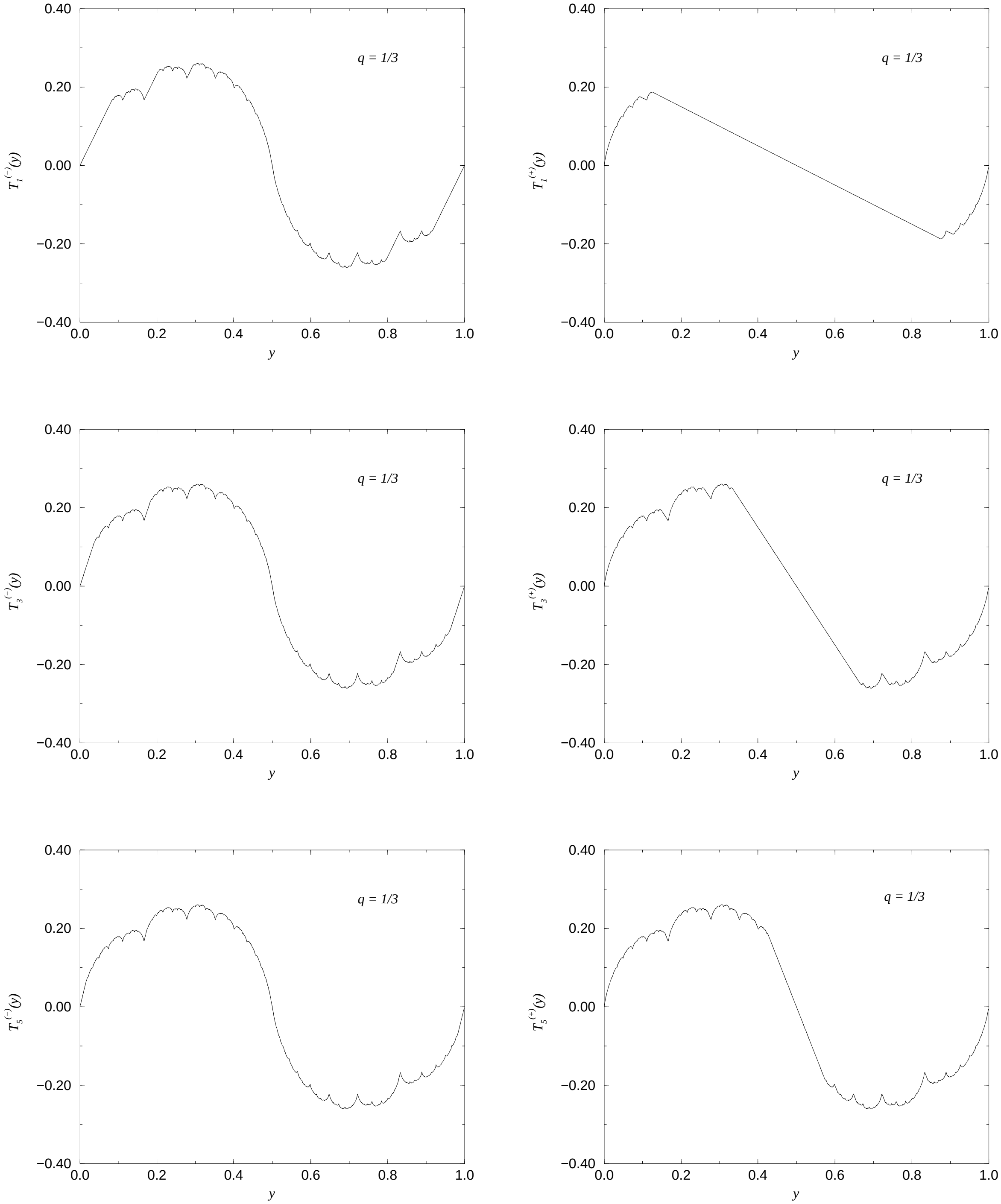

FIG. 6. The incomplete Takagi functions $T_n^{(\pm)}(y)$, Eq. (29), on the right and left respectively, for $q = 1/3$ and $n = 1, 3, 5$, from top to bottom. The size of the chain is $L = 100$.

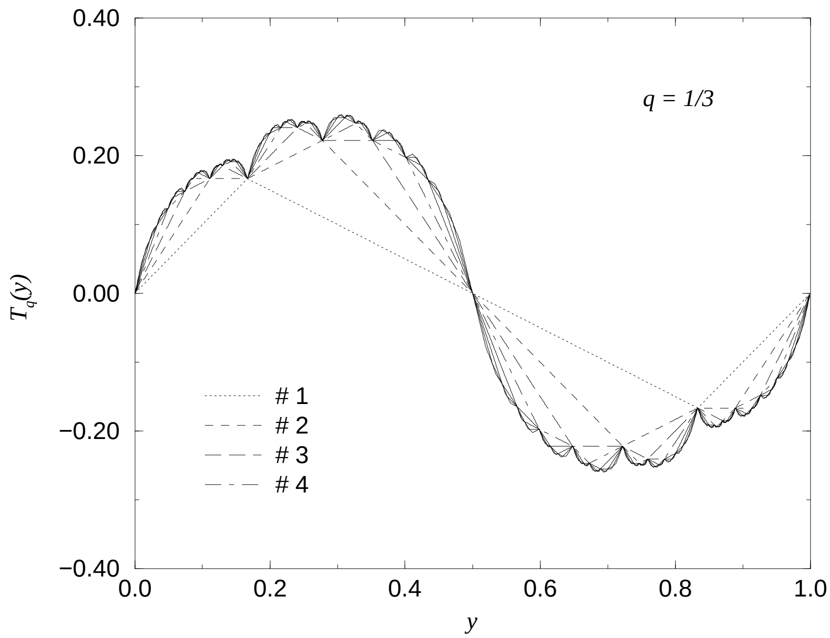

FIG. 7. A recursive computation of the generalized Takagi function T_q, Eq. (31) for $q = 1/3$. The legend indicates the numbers of the first four iterates. A total of 10 iterates are displayed.

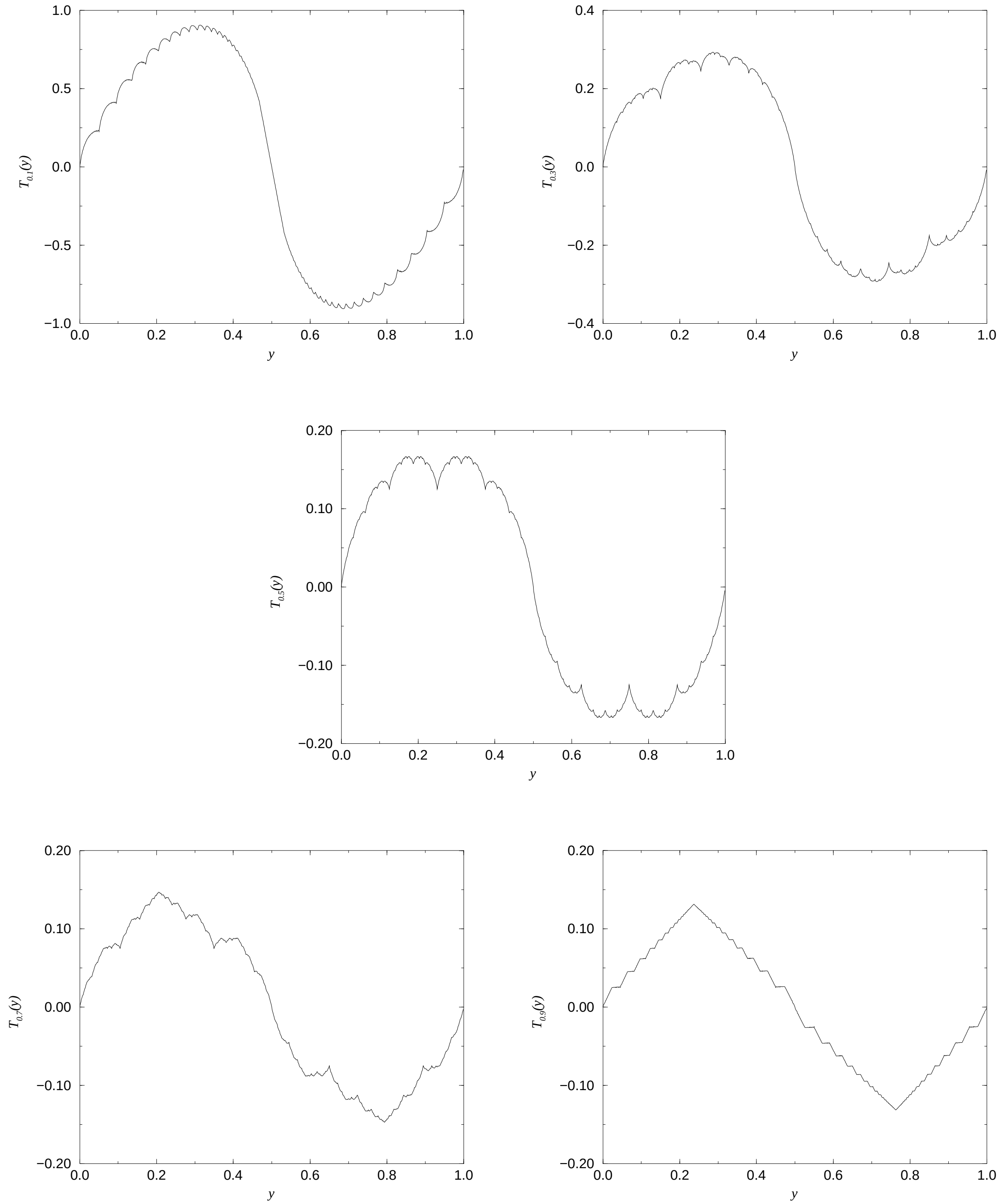

FIG. 8. The functions $T_q(y)$, Eq. (31), displayed for five different values of $q = 0.1, \ldots, 0.9$, from top to bottom and left to right. The figure in the center, $q = 0.5$, corresponds to the symmetric case.